# No Black Hole Theorem in Three-Dimensional Gravity

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A common property of known black hole solutions in (2+1)-dimensional gravity is that they require a negative cosmological constant. In this letter, it is shown that a (2+1)-dimensional gravity theory which satisfies the dominant energy condition forbids the existence of a black hole to explain the above situation.

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The (2+1)-dimensional theory provides us with one of useful approaches to more complicated (3+1)-dimensional classical gravity or conceptual problems in quantum gravity [1]. At first sight, the (2+1)-dimensional gravity looks trivial. In particular, the vacuum Einstein equation implies that the space-time is locally flat, corresponding to the absence of the gravitational radiation (Weyl tensor) in three dimensions. However, the local distribution of matter fields has a global effect on the outer empty space; for instance, the gravitational field of a point particle is described by a conical space with its deficit angle corresponding to the mass of the particle [2], which causes the gravitational lens effect. One should also note that the triviality of local geometry does not necessarily imply the triviality of the theory itself; namely, the topological degrees of freedom plays an important role in the theory of gravitation [3,4]. The triviality of local geometry in the (2+1)-gravity theory holds even if the cosmological term is taken into account. The Einstein space is simply a space of constant curvature in three dimensions, so that educated relativists would not imagine that there is a black hole solution in this theory until in 1992 Bañados *et al.* show that there actually exists a black hole in the locally anti-de Sitter space [5,6]. This black hole space-time, called BTZ black hole, is obtained by identifying certain points of (the covering manifold of) the anti-de Sitter space. A different identification makes a space-time representing the BTZ black hole in a closed universe [9], multiple BTZ black holes [10] or a creation of the BTZ black hole [11]. The BTZ black hole is characterized by the mass, angular momentum and cosmological constant, and has almost all features of the Kerr-anti-de Sitter black hole in the conventional four-dimensional Einstein gravity. The BTZ black hole was shown to be also the solution of a low energy string theory [7,8].

Since the discovery of the BTZ black hole, a number of authors have attempted to find a black hole solution in various theories in (2+1)-dimensions. Black holes in topologically massive gravity [12] with the negative cosmological constant were found by Nutku [13]. In Einstein-Maxwell-$\Lambda$ system, a static (non-rotating) charged black hole had been already noted in the original paper by Bañados *et al.* [5]. Clément [14] generated from the charged BTZ black hole a class of rotating charged black holes. Though rotating solutions in Einstein-Maxwell-$\Lambda$ theory seem to have infinite total mass and angular momentum, these divergences may be cured by adding a Chern-Simons term to the action [14]. Black holes with a dilaton field have been discussed by many authors. In Brans-Dicke theory, Sa *et al.* found black hole solutions [16,17], and their properties were extensively studied for different Brans-Dicke parameters. Black holes in Einstein-Maxwell-dilaton-$\Lambda$ theory were obtained by Chan and Mann in non-rotating [18] and rotating [19] cases. Other families were given by Koikawa *et al.* [20] and by Fernando [21]. Chen [22] also derived rotating black hole solutions in this theory by means of the duality transformation in the equivalent non-linear $\sigma$-model. Black holes coupled to a topological matter field [23], conformal scalar field [24], Yang-Mills field [25], Born-Infeld field [26] *etc.* were also discussed.

Thus, many black hole solutions are known Here, it might be interesting to note that all the black hole solutions listed above require a negative cosmological constant, otherwise a certain kind of energy conditions is violated. A typical example might be the BTZ black hole. As already mentioned, the BTZ black hole may be constructed by making identifications in the anti-de Sitter space. We may also consider a similar construction in the de Sitter space. In this case, a natural procedure might be identifying two geodesic circles in each Poincaré disk associated with the open chart of the de Sitter space. The resultant space-time represents an inflating universe rather than a black hole. The absence of black hole in this example might be due to the difference in the causal structure of conformal infinity [27].

The purpose of this letter is to give a reason for this situation. In particular, we will be able to answer the question: *"Why the BTZ black hole requires a negative cosmological constant?"* In the following, we consider the possibility of the existence of a black hole (in the sense of an apparent horizon) in three-dimensional space-time with the procedure given by Hawking [28] in terms of the spin-coefficient formalism [29].

Let $(M, g)$ be a three-dimensional space-time and let

$\Sigma$ be a space-like hypersurface in $M$. Suppose that $\Sigma$ contains outer trapped surfaces, then there will be an apparent horizon $H$ which is defined to be the outer boundary of the trapped region in $\Sigma$, where the notion "outer" is assumed to be well-defined as in the case of the asymptotically flat (or anti-de Sitter) space-time. We also assume that the apparent horizon $H$ is a smooth closed curve in $\Sigma$. Let $m$ be a unit tangent vector of $H$, and let $n$ and $n'$ be future directed out-going and in-going null vectors orthogonal to $H$, respectively, such that $g(n,n')=1$. The vectors $n$ and $n'$ are arranged such that $n-n'$ lies in $\Sigma$, which is always possible by means of the boost transformation $n\mapsto a^2 n$, $n'\mapsto a^{-2}n'$ by some positive function $a$. Let us consider a local deformation of $H$ within $\Sigma$ outside the trapped region generated by a vector field $X=e^f(n-n')$ with some smooth function $f$. Accordingly, the null triad $\{n,n',m\}$ is extended such that the normalization $g(n,n')=-g(m,m)=1$, $g(n,n)=g(n',n')=g(n,m)=g(n',m)=0$ is preserved and that $m$ is tangent to each deformed $H$. Then, since $X$ and $Y=e^h m$ form holonomic base vectors on $\Sigma$ for some function $h$, $n$ and $n'$ are propagated such that

$$\delta f=\kappa-\tau+\beta=\kappa'-\tau'-\beta, \tag{1}$$

where Ricci rotation coefficients

$$\begin{aligned}&\kappa=g(m,Dn),\quad \tau=g(m,D'n),\quad \beta=g(n',\delta n),\\ &\kappa'=g(m,D'n'),\quad \tau'=g(m,Dn')\end{aligned} \tag{2}$$

and the differential operators

$$D=\nabla_n,\quad D'=\nabla_{n'},\quad \delta=\nabla_m \tag{3}$$

are defined following the spin-coefficient formalism in four space-time dimensions [29]. The convergence of light rays emitted outward from each deformed $H$ is measured by the quantity

$$\rho=g(m,\delta n). \tag{4}$$

In particular, $\rho=0$ holds on $H$ since $H$ will be a marginally trapped surface. The change in $\rho$ along $X$ is derived by the following equations

$$D\rho-\delta\kappa=(\epsilon+\rho)\rho-(2\beta+\tau+\tau')\kappa+\phi_{++}, \tag{5}$$

$$D'\rho-\delta\tau=-\epsilon'\rho-\kappa\kappa'-\tau^2+\rho\rho'-\phi_{+-}-\Pi, \tag{6}$$

where

$$\begin{aligned}&\epsilon=g(n',Dn),\quad \epsilon'=g(n,D'n'),\quad \rho'=g(m,\delta n'),\\ &\phi_{++}=\phi(n,n),\quad \phi_{+-}=\phi(n,n'),\quad \Pi=R/6\end{aligned} \tag{7}$$

with the trace-free part of the Ricci tensor $\phi=-\mathrm{Ric}+(R/3)g$. Subtracting Eq. (6) from Eq. (5), we obtain the equation

$$\begin{aligned}e^{-f}\mathcal{L}_X\rho&=\delta(\kappa-\tau)-(2\beta+\tau+\tau')\kappa+\kappa\kappa'+\tau^2\\ &\quad+\phi_{++}+\phi_{+-}+\Pi\\ &=\delta(\delta f-\beta)+(\kappa-\tau)^2+\phi_{++}+\phi_{+-}+\Pi\end{aligned} \tag{8}$$

on $H$, where Eq. (1) has been used. Now suppose that there is a positive cosmological constant $\Lambda>0$ and that the stress-energy tensor $T$ satisfies *the dominant energy condition: (i)* $T(W,W)\geq 0$*, and (ii)* $T(W)$ *is non-space-like, for every time-like vector* $W$. Then, the Einstein equation $\mathrm{Ric}-(R/2)g+\Lambda g=-8\pi T$ leads to the inequalities

$$\phi_{++}\geq 0,\quad \phi_{+-}+\Pi>0. \tag{9}$$

The term $\delta(\delta f-\beta)$ in the last line of the Eq. (8) can be made zero by appropriately choosing the function $f$; in fact, parametrizing $H$ by the proper length $s\in[0,\mathrm{Length}(H))$, such a function $f$ can be explicitely written as

$$f=\int^s\beta\mathrm{d}s-\left(\frac{\oint\beta\mathrm{d}s}{\oint\mathrm{d}s}\right)s. \tag{10}$$

Then, the last line of the Eq. (8) is positive definite, $\mathcal{L}_X\rho>0$. This implies that there is an outer trapped surface outside $H$, which contradicts the assumption that $H$ is the outer boundary of such surfaces. Hence, we obtain the following no black hole theorem:

**Theorem 1** *Let* $(M,g)$ *be a three-dimensional space-time subject to the Einstein equation* $\mathrm{Ric}-(R/2)g+\Lambda g=-8\pi T$ *with* $\Lambda>0$*. If the stress-energy tensor* $T$ *satisfies the dominant energy condition, then* $(M,g)$ *contains no apparent horizons.*

This explains why black hole solutions require a negative cosmological constant. Strictly speaking, we can only say that there is no non-degenerate apparent horizon ($\rho=0$, $\mathcal{L}_X\rho\neq 0$) in the case of $\Lambda=0$, however, the presence of matter fields such as the dilaton or Maxwell field will exclude even degenerate horizons.

Thus, a black hole in (2+1)-gravity requires negative energy such as a negative cosmological constant. This implies the breakdown of the predictability in certain three-dimensional theories. As in four space-time dimensions, we may consider the Oppenheimer-Snyder model of the gravitational collapse. The homogeneous disk of dust will collapse to a central point and a naked conical singularity will be left. This picture of gravitational collapses will remain unchanged unless the negative cosmological constant is added. Even in the case of the non-symmetric gravitational collapse of gauge fields or scalar fields, there will not form a black hole, so that when a singularity is formed, such a singularity will be naked.

We have disscussed the existence problem of apparent horizons, while the black hole is often defined by the event horizon. Since the theorem 1 relies on the local analysis, we cannot argue the global structure of space-time such as an event horizon. An important exception is the stationary case; we can replace "apparent horizons" with "stationary event horizons" in the theorem 1, since it is known that these coincide in this case.

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